N O T I C E

THIS DOCUMENT HAS BEEN REPRODUCED FROM MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED IN THE INTEREST OF MAKING AVAILABLE AS MUCH INFORMATION AS POSSIBLE

NASA TECHNICAL MEMORANDUM NASA TM-75800

A MODEL THEORY FOR THE FIBROUS ABSORBER, PART 1: REGULAR FIBRE ARRANGEMENTS

F. P. Mechel

(NASA-TM-75800) A MODEL THEORY FOR THE FIBROUS ABSORBER. PART 1: REGULAR FIBRE ARRANGEMENTS (National Aeronautics and Space Administration) 33 p HC A03/MF A01 CSCL 20A

N80-21101

Unclas

G3/71 47631

Translation of "Eine Modelltheorie zum Faserabsorber Teil 1: Regulare Faseranordnung", Acustica Vol. 36, Issue No. 2, 1976/77, pp. 53 - 64.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. MARCH 1980

STANDARD TITLE PAGE

1. Report No. NASA TM-75800	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A MODEL THEORY FOR THE FIBROUS ABSORBER, PART 1: REGULAR FIBRE ARRANGEMENTS		5. Report Date March 1980
		6. Performing Organization Code
7. Author(s) F. P. Mechel, University of the Saarland Acoustics Laboratory, Saarbrucken		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		11. Contract or Grant No. NASw- 3199
		13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of "Eine Modelltheorie zum Faserabsorber Teil 1: Regulare Faseranordnung", Acustica Vol. 36, Issue No. 2, 1976/77, pp. 53-64

16. Abstract

The axial propagation of sound waves in a model consisting of parallel fibres will be calculated. The viscous forces and the thermal conduction are taken into account. This will lead to viscous waves and to thermal waves besides the usual acoustic compression wave. The potential function for the total field near a fibre will be treated as the superposition of the radiated field from the fibre itself and of the scattered fields from all the other fibres. The explicit field equations for a regular square fibre arrangement will be derived and the influence of the order of symmetry of the arrangement will be discussed. This will lead to simplifications in the field equations and to field equations for the case of a homogeneous fibre distribution.

17. Key Words (Selected by Author(s))		18. Distribution Statement Unclassified unlimited.	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 33	22. Price

ORIGINAL PAGE IS OF POOR QUALITY

NASA-HQ

A	Vector potential of the viscosity wave
a	Fiber radius
c_0	Adiabatic speed of sound
d	Fiber separation (in the model of a regular absorber)
$E_{\varrho,\alpha,\nu}$	Amplitudes of wave types in air
F_l	Amplitude of temperature wave inside the fiber
G	Internal wave conductance of the absorber
$H_n^{(2)}$	Hankel functions of n-th order, second type
h	Porosity of the absorber
$h^* = 1 - h$	Fill factor
$k_{0,\alpha,\nu,l}$	Wave numbers of wave types
N	Fiber density (number of fibers per surface unit)
p_0	Stationary pressure fraction
p_1	Alternating pressure
R	Radius of an elementary cell (fiber in the tube)
r, φ, z	Cylinder coordinates
$S_l(\beta)$	See equation (151) /sic/
T_0	Stationary temperature of the air
T_1	Alternating temperature of air
T_{01}	Stationary temperature of the fiber
T_{11}	Alternating temperature of the fiber
v	Speed of sound
W	Internal wave resistance of the absorber
Z, Z^*	Surface impedance (with *: standardized to $\varrho_0 c_0$).
Z_n	General cylinder function of n-th order
$\alpha = \Lambda/(\varrho_0 c_p)$	Temperature conductance coefficient of air
α_1	Temperature conductance coefficient of the fiber material
Γ, Γ^*	Propagation constant (with *: standardized to ω/c_0),)
$\gamma = c_p/c_v$	Adiabatic exponent
η	Dynamic viscosity
$\Phi = \Phi_\varrho + \Phi_\alpha$	Scalar potential
Φ_ϱ	Scalar potential of the compressional wave
Φ_α	Scalar potential of the temperature wave
Λ	Thermal conductance of air
Λ_1	Thermal conductance of the fibre material
$\nu = \eta/\varrho_0$	Kinematic viscosity
$\omega = 2\pi f$	Angular frequency
ϱ_0	Stationary density of air
ϱ_1	Alternating density of air
σ	Specific flow resistance
$\hat{\sigma}$	Flow resistance

Indices:

α	Temperature wave in air
i	Temperature wave of the fiber
ν	Viscosity wave
ϱ	Compressional wave
′ (prime):	emitting field
″ (double prime):	scattering field, constant field
^ (circumflex):	amplitudes determined by model geometry

A MODEL THEORY FOR THE FIBROUS ABSORBER,
PART 1: REGULAR FIBRE ARRANGEMENTS

F. P. Mechel

University of the Saarland Acoustics Laboratory, Saarbrucken

1. Introduction /53*

Most sound absorbers are fiber absorbers: the absorption materials consist of loosely interconnected fibers. The fibres themselves are usually glass or mineral fibers, but also organic fibers and--for special applications--metal fibers are used.

To describe sound propagation in these fiber materials, at present there are practically two theories. The first, which we will call the theory of quasi-homogeneous absorber, describes the absorber as a homogeneous, isotropic medium and considers the losses in sound energy in the absorber by a flow resistance and tries to describe the structure by a "structure factor" in the force equation. The second theory replaces the absorber by a model of a bundle of parallel tubes (Rayleigh model) with sound resistant, infinite heat conducting walls. Both theories have their obvious and known deficiencies.

In an earlier study /1/ it was found which of the two theories was best suited for further development. It turned out that the theory of the quasi-homogeneous absorber leads to internal contradictions in addition to the disadvantage of frequency dependent material constants which cannot be eliminated by a simple change in the theory. Conversely, the theory of the tube model appears to correctly reproduce the physical processes of sound propagation in the fiber absorber, but suffers by having excessive deviation between model and structure and real absorber structure.

Therefore it was suggested to apply the principles of the theory of the Rayleigh model, namely the solution of the most exact and com- /54
plete differential equations to a model better adapted to the real absorber structure.

*Numbers in the margin indicate pagination in the foreign text.

The model used here consists of parallel fibers of the same material just like the real absorber. The solutions of the complete differential equations (i.e. under consideration of air viscosity between the fibers and thermal conductance of both the air and fibers) are given. The sound wave should propagate in the direction of the fibers.

Initially omitted properties of a real abosrber are:

1) scattering of fiber radii about an average value
2) melt beads and adhesive globs
3) statistic orientation of the fibers in a plane.

While work with this theory was underway, the excellent papers by Attenborough et al. /2/, /3/ appeared. There, the theory of multiple scattering from cylinders was applied to the fiber absorber. This is a method tailored specifically to the absorber attacked perpendicular to the fiber direction.

Since we are assuming a sound propagation parallel to the fiber orientation, the model described here represents a supplement to the work of Attenborough. In addition, the calculation presented here represents a logical continuance of the Rayleigh model. In the Rayleigh model the sound propagates axis-parallel in the tube. We eliminate the main error of this model, namely the tube, whose wall consists of fibre material, and study instead the axis-parallel acoustic propagation in a fiber bundle.

In an initial equation for this model we assume a regular arrangement of fibers in a quadratic grid. This is certainly a flaw in the model by comparison to a real absorber. Its effect is even more ominous since we frequently had to use symmetry properties of this arrangement in the course of the calculation. The method of calculation shows which changes occur in the result if instead of quadratic symmetry, a symmetry of higher order is postulated.

On the basis of these results a simplified model is discussed in a subsequent part of this work (part II). Here, each fiber is surrounded by a imaginary cylindrical symmetry surface, i.e. we use a symmetry of infinite order. As with the Rayleigh model, we only need to find the solutions in this type of cylinder with one fiber on the axis.

2. Basic Equations /55

The scalar quantities of state of the sound field, namely, pressure p , density ϱ , temperature T are split apart into stationary fractions with the index zero and into chronologically variable fractions having the index one in accordance with exp ($j\omega t$). For the sound field quantities the usual linearization assumptions are made that their squares are negligible compared to linear terms. We then have:

the equation of force (Navier-Stokes):

$$\frac{\partial v}{\partial t} + \frac{1}{\varrho_0} \operatorname{grad} p_1 - \nu \Delta v - \tfrac{1}{3} \nu \operatorname{grad} \operatorname{div} v = 0, \tag{1}$$

the continuity equation:

$$\frac{\partial \varrho_1}{\partial t} + \varrho_0 \operatorname{div} v = 0, \tag{2}$$

the equation of thermal energy:

$$\frac{\partial T_1}{\partial t} + (\gamma - 1) T_0 \operatorname{div} v - \alpha \Delta T_1 = 0, \tag{3}$$

the equation of state:

$$\frac{p_1}{p_0} - \frac{\varrho_1}{\varrho_0} - \frac{T_1}{T_0} = 0 \tag{4}$$

and the equation of thermal conductance in the fiber:

$$\frac{\partial T_{11}}{\partial t} - \alpha_1 \Delta T_{11} = 0. \tag{5}$$

In equation 1 we neglect convective acceleration and in equation 3, thermal transport by radiation.

3. Potential Functions and Wave Equations

The standard potential functions for the acoustic field are defined by:

$$v = -\operatorname{grad}\Phi + \operatorname{rot}A \tag{6}$$

with the secondard condition

$$\operatorname{div}A = 0. \tag{7}$$

After substituting into equation 1 under consideration of the time law exp ($j\omega t$) and the identity:

$$\Delta = \operatorname{grad}\operatorname{div} - \operatorname{rot}\operatorname{rot}$$

rot = red

we obtain:

$$-\operatorname{grad}\left[j\omega\Phi - \frac{p_1}{\varrho_0} - \frac{4}{3}\nu\Delta\Phi\right] + \operatorname{rot}\left[j\omega A - \nu\Delta A\right] = 0. \tag{8}$$

Since both summands are independent of each other, they must disappear individually and we have:

$$j\omega\Phi - \frac{p_1}{\varrho_0} - \frac{4}{3}\nu\Delta\Phi = 0, \tag{9}$$

$$j\omega A - \nu\Delta A = 0. \tag{10}$$

Equation 10 comprises the first wave equation, namely for the viscosity wave:

$$(\Delta + k_\nu^2)A = 0 \tag{11}$$

with:

$$k_\nu^2 = -j\frac{\omega}{\nu}.$$

Substituting equation 6 into equation 2 gives:

$$\frac{\varrho_1}{\varrho_0} = \frac{\Delta\Phi}{j\omega} \tag{12}$$

and substituting equation 6 into equation 3 gives:

$$j\omega T_1 - (\gamma - 1)\, T_0 \Delta\Phi - \varkappa \Delta T_1 = 0\,. \tag{13}$$

In order to obtain a wave equation for Φ, we must eliminate the middle term in equation 9. Substituting equation 12 into equation 4 gives:

$$\frac{p_1}{\varrho_0} = \frac{p_0}{\varrho_0}\left(\frac{\Delta\Phi}{j\omega} + \frac{T_1}{T_0}\right). \tag{14}$$

Substituting equation 14 into equation 9 gives:

$$j\omega\Phi - \frac{p_0}{\varrho_0}\left(\frac{\Delta\Phi}{j\omega} + \frac{T_1}{T_0}\right) - \frac{4}{3}\nu\Delta\Phi = 0\,. \tag{15}$$

Substituting $\Delta\Phi$ from equation 14 into equation 9:

$$j\omega\Delta\Phi - \frac{p_0}{\varrho_0}\left(\frac{\Delta^2\Phi}{j\omega} + \frac{j\omega}{\alpha}\frac{T_1}{T_0} - \frac{\gamma-1}{\alpha}\Delta\Phi\right) - \frac{4}{3}\nu\Delta^2\Phi = 0\,. \tag{16}$$

Equation 16 minus equation 15 multiplied by $j\omega/\alpha$ results in: /56

$$\left\{j\alpha\left(\frac{c_0^2}{\gamma\omega} + \frac{4}{3}j\nu\right)\Delta^2 + \omega\left[\frac{c_0^2}{\omega} + j\left(\alpha + \frac{4}{3}\nu\right)\right]\Delta + \omega^2\right\}\Phi = 0 \tag{17}$$

with

$$c_0^2 = \gamma p_0/\varrho_0\,. \tag{18}$$

According to Lord Rayleigh, equation 17 can be written as a product of two wave equations

$$(\Delta + k_\varrho^2)(\Delta + k_\alpha^2)\Phi = 0\,. \tag{19}$$

Here, k_ϱ^2, k_α^2 are the negative solutions of the quadratic equation by Δ, obtained by equating the brackets in equation 17 with zero. From this we have:

$$\left.\begin{matrix} k_\varrho^2 \\ k_\alpha^2 \end{matrix}\right\} = j\omega\,\frac{-\left[\frac{c_0^2}{\omega} + j\left(\alpha + \frac{4}{3}\nu\right)\right] \pm \sqrt{\left[\frac{c_0^2}{\omega} + j\left(\alpha + \frac{4}{3}\nu\right)\right]^2 - 4j\alpha\left(\frac{c_0^2}{\gamma\omega} + \frac{4}{3}j\nu\right)}}{2\alpha\left(\frac{c_0^2}{\gamma\omega} + \frac{4}{3}j\nu\right)}\,. \tag{20}$$

Equation 19 is solved if we set:

$$\Phi = \Phi_\varrho + \Phi_\alpha, \tag{21}$$

$$(\Delta + k_\varrho^2)\Phi_\varrho = 0, \tag{22}$$

$$(\Delta + k_\alpha^2)\Phi_\alpha = 0. \tag{23}$$

If we consider the order of magnitude of the constants, we see that by approximation we only need to consider the first term in parenthesis in front of Δ and Δ^2 in equation 17. From this we have:

$$\left.\begin{matrix} k_\varrho^2 \\ k_\alpha^2 \end{matrix}\right\} = \mathrm{j}\frac{\gamma\omega}{2\alpha}\left(-1 \pm \sqrt{1 - 4\mathrm{j}\frac{\alpha\omega}{\gamma c_0^2}}\right)$$

and if we extract the root:

$$k_\varrho^2 \approx (\omega/c_0)^2, \quad k_\alpha^2 \approx -\mathrm{j}\gamma\omega/\alpha. \tag{24}$$

k_ϱ is the wave number of the standarized sound wave in air, that is, a compressional wave; k_α, is the wave number of a temperature wave.

Accordingly, three wave types appear in the absorber: the normal compression wave defined by Φ_ϱ , a temperature wave described by Φ_α and a viscosity wave with vector potential A .

4. Quantities of State

It is our objective to find solutions for the potential functions. In order to describe the acoustic field the acoustic field quantities must be expressed by these potential functions.

The speed of sound is obtained from equation 6 and 21:

$$v = -\operatorname{grad}(\Phi_\varrho + \Phi_\alpha) + \operatorname{rot} A. \tag{25}$$

The relative change in density is obtained from equation 12 with equation 22 and 23 as:

$$\frac{\varrho_1}{\varrho_0} = -\frac{\mathrm{j}}{\omega}[k_\varrho^2\Phi_\varrho + k_\alpha^2\Phi_\alpha]. \tag{26}$$

From equation 9 we obtain the relative sound pressure as:

$$\frac{p_1}{p_0} = H_\varrho\Phi_\varrho + H_\alpha\Phi_\alpha \tag{27}$$

with the coefficents:

$$H_{\varrho,\alpha} = \frac{\gamma}{c_0^2}\left(\mathrm{j}\omega + \frac{4}{3}\nu k_{\varrho,\alpha}^2\right). \tag{28}$$

As in equation 28, expressions with the indices ϱ,α will appear below. We intend to use these as abbreviations for two equations, each of which would otherwise have only one index. Therefore, equation 28 stands for:

$$H_\varrho = \frac{\gamma}{c_0^2}\left(\mathrm{j}\omega + \frac{4}{3}\nu k_\varrho^2\right) \quad \text{and}$$

$$H_\alpha = \frac{\gamma}{c_0^2}\left(\mathrm{j}\omega + \frac{4}{3}\nu k_\alpha^2\right).$$

From equation 4 we then obtain the relative temperature change as:

$$\frac{T_1}{T_0} = \Theta_\varrho \Phi_\varrho + \Theta_\alpha \Phi_\alpha \tag{29}$$

with the coefficients :

$$\Theta_{\varrho,\alpha} = \frac{4}{3}\frac{\gamma\nu}{c_0^2}k_{\varrho,\alpha}^2 + \mathrm{j}\left(\omega\frac{\gamma}{c_0^2} - \frac{k_{\varrho,\alpha}^2}{\omega}\right). \tag{30}$$

The following discussion where the explicit dependence on the kinematic viscosity ν is eliminated also pertains to the coefficients $H_{\varrho,\alpha}$ and $\Theta_{\varrho,\alpha}$:

$$H_{\varrho,\alpha} = \frac{\mathrm{j}k_{\varrho,\alpha}^2}{\omega}\,\frac{\gamma\omega - \mathrm{j}\alpha k_{\varrho,\alpha}^2}{\omega - \mathrm{j}\alpha k_{\varrho,\alpha}^2}, \tag{31}$$

$$\Theta_{\varrho,\alpha} = \frac{\mathrm{j}(\gamma - 1)k_{\varrho,\alpha}^2}{\omega - \mathrm{j}\alpha k_{\varrho,\alpha}^2}. \tag{32}$$

5. Boundary Conditions /57

The emission conditions and certain symmetry conditions, depending on the model, must be met. The following conditions must be met at the fiber surfaces by the resulting mathmatical descriptions of the potentials: disappearance of the speed with all components, equality of temperatures of air and fibers, equality of radial thermal flux density in air and fiber at the fiber surface, i.e.:

$$\left.\begin{aligned} &v_r, v_\varphi, v_z = 0 && (33)\\ &T_1 = T_{11} && (34)\\ &\Lambda\frac{\partial}{\partial r}T_1 = \Lambda_1\frac{\partial}{\partial r}T_{11} && (35)\end{aligned}\right\}\ \text{für}^{*}\ r = a.$$

* for

As a simplified thermal boundary condition we studied the disappearance of alternating temperature at the fiber surface, i.e.:

$$T_1(r = a) = 0.$$

Although the boundary conditions to be met by the individual fibers are always the same, with the individual model there result various descriptions of the potentials (e.g. due to different symmetry properties) which are to meet the above boundary conditions.

6. Individual Fiber

We first consider an isolated fiber. Let it be embedded in a cylindrical coordinate system (r, φ, z), with the z-axis coinciding with the fiber axis. Let us call the fiber radius a. As the time function we select $\exp(j\omega t)$, as the z-function $\exp(-\Gamma z)$. From the geometry of the arrangement, we have immediate axial symmetry, i.e. $\partial/\partial\varphi = 0$. From equations 22 and 23 we have:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \Gamma^2 + k^2_{\varrho,\alpha}\right)\Phi_{\varrho,\alpha} = 0. \qquad (36)$$

If we set:

$$A = (A_r, A_\varphi, A_z), \qquad (37)$$

Then because of

$$\partial/\partial\varphi = 0$$

we have:

$$v_\varphi = \mathrm{rot}_\varphi A = \frac{\partial}{\partial z} A_r - \frac{\partial}{\partial r} A_z = 0 \qquad (38)$$

and because of $\mathrm{div}\, A = 0$:

$$\frac{\partial}{\partial r} A_r + \frac{1}{r}\frac{\partial}{\partial\varphi} A_\varphi + \frac{\partial}{\partial z} A_z = 0. \qquad (39)$$

Applying $\partial/\partial z$ to equation 38 and $\partial/\partial r$ to equation 39 gives:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) A_r = (\Delta A)_r = 0$$

And with equation 11 there results:

$$k_\nu^2 A_r = 0, \text{ i.e. } A_r = 0.$$

From equation 38 and 39 we have:

$$\frac{\partial}{\partial r} A_z = 0$$

and

$$\frac{\partial}{\partial z} A_z = 0,$$

that is

$$A_z = 0.$$

Therefore, of the vector potential there remains only the component A_φ for which the wave equation applies:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \Gamma^2 + k_\nu^2\right) A_\varphi = 0. \quad (40)$$

As solutions of equation 36 or 40 , Hankel functions of zero or first order are used. Because of the emission conditions only the second type of Hankel function come into consideration if we choose the imaginary fraction of the cross-propagation constant to be negative:

$$\mathrm{Im}(r) < 0. \qquad (41)$$

Thus we have:

$$\Phi_\varrho(r,z) = E_\varrho e^{-\Gamma z} H_0^{(2)}(r_\varrho r), \qquad (42)$$

$$\Phi_\alpha(r,z) = E_\alpha e^{-\Gamma z} H_0^{(2)}(r_\alpha r), \qquad (43)$$

$$A_\varphi(r,z) = E_\nu e^{-\Gamma z} H_1^{(2)}(r_\nu r) \qquad (44)$$

with

$$r^2_{\varrho,\alpha,\nu} = \Gamma^2 + k^2_{\varrho,\alpha,\nu}. \qquad (45)$$

With these equations the boundary conditions are met at the fiber surface.

As a result of axial symmetry, $v_\varphi(r = a) = 0$ is also true. From the other conditions there result the following equations:

$$v_r(r = a) = 0:$$

$$\sum_{\varrho,\alpha} E_\varrho r_\varrho H_1(r_\varrho a) + E_\nu \Gamma H_1(r_\nu a) = 0. \qquad (46)$$

The sum $\sum\limits_{\varrho,\alpha}$ in equation 46 is an abbreviation for the sum of two terms. The first summand is the one behind the summation sign, the second is obtained from the first if we replace the index ϱ by the index α.

From the condition v_z $(r = z) = 0$ we obtain: /58

$$\sum_{\varrho,\alpha} E_\varrho \Gamma H_0(\varepsilon_\varrho a) + E_\nu \varepsilon_\nu H_0(\varepsilon_\nu a) = 0. \qquad (47)$$

From the equality of the internal and external temperature at the fiber surface we obtain:

$$\sum_{\varrho,\alpha} E_\varrho \Theta_\varrho H_0(\varepsilon_\varrho a) - F_1 J_0(\varepsilon_1 a) = 0 \qquad (48)$$

and from the equality of the radial heat flux it follows:

$$\sum_{\varrho,\alpha} E_\varrho \Theta_\varrho \varepsilon_\varrho H_1(\varepsilon_\varrho a) - F_1 \frac{\Lambda_1}{\Lambda} \varepsilon_1 J_1(\varepsilon_1 a) = 0. \qquad (49)$$

From equation 5 we have the following expression for the wave equation:

$$(\Delta + k_1^2) T_{11} = 0 \qquad (50)$$

with:

$$k_1^2 = -j\omega/\varkappa_1 \qquad (51)$$

We obtain the solution equation:

$$\frac{T_{11}(r,z)}{T_0} = F_1 e^{-\Gamma z} J_0(\varepsilon_1 r) \qquad (52)$$

with:

$$\varepsilon_1^2 = k_1^2 + \Gamma^2. \qquad (53)$$

To determine the propagation constants Γ the determinant of the equation system (46) to (49) is eliminated for the amplitudes E_ϱ, E_α, E_ν, F_1. This system of equations can now be written for the sake of clarity as follows:

$$\sum_{\varrho,\alpha} E_\varrho \varepsilon_\varrho H_1(\varepsilon_\varrho a) + E_\nu \Gamma H_1(\varepsilon_\nu a) = 0,$$
$$\sum_{\varrho,\alpha} E_\varrho \Gamma H_0(\varepsilon_\varrho a) + E_\nu \varepsilon_\nu H_0(\varepsilon_\nu a) = 0,$$
$$\sum_{\varrho,\alpha} E_\varrho \Theta_\varrho H_0(\varepsilon_\varrho a) - F_1 J_0(\varepsilon_1 a) = 0,$$
$$\sum_{\varrho,\alpha} E_\varrho \Theta_\varrho \varepsilon_\varrho H_1(\varepsilon_\varrho a) - F_1 \frac{\Lambda_1}{\Lambda} \varepsilon_1 J_1(\varepsilon_1 a) = 0.$$

The numerical equation shows that for fiber radii as used in real absorbers, the propagation constant of an individual fiber is not affected. But in order to obtain calcuable deviations in propagation constant from the free field wave number, fiber radii of at least 20 cm must be assumed.

The calculation performed here permits us in future models to separate the field of an individual fiber from the total field in the absorber.

7. Quadratic Arrangement of the Fibers

7.1 Geometry

The geometric conditions are illustrated in Figure 1. We call the fiber radius a, the distance of the axes of two neighboring fibers is d.

In the cartesian coordinate system(x, y) the fiber axes have coordinates (md, nd), whereby m, n are whole numbers. The individual fibers are identified by the indices m, n. In the cylinder coordinate system (r, φ), the fiber (m, n) has coodinate (R_{mn}, ψ_{mn}).

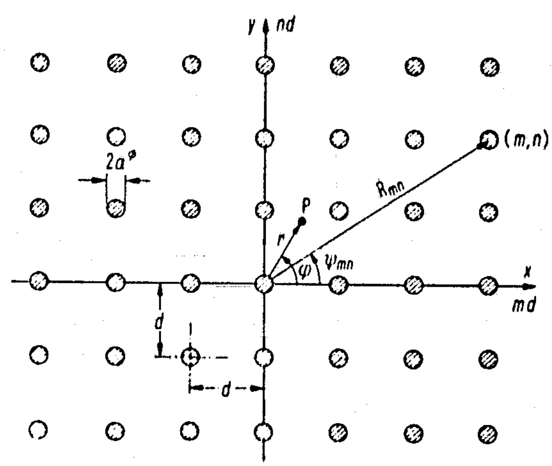

Figure 1: Coordinates of the fibers and plotted point P.

The potentials are periodic in φ by $\pi/2$:

$$\Phi(r, \varphi, z) = \Phi(r, \varphi + \pi/2, z).$$

The potentials are symmetric at $\varphi = 0, \pi/4$.
In the treatment of this model we assume that the field about a fibre is composed of a fraction as would be generated by the individual fiber (emitted field) and a fraction composed of the sum of the emitted fields of all other fibers (scattering field). The emitting field is assumed to be axial symmetric, the scattering field is dependent on the angle.

From the shape of the emitting part of a fiber (see equation 42 to 44) there results that this field is inconstant on the axis of this fiber. The scattering field caused by the other fibers is naturally constant and therefore instead of the concept "emitting field" and "scattering field" we also use the expressions "inconstant field" and "constant field".

7.2 Determination of the Scalar Potential

For the scalar potential we use the wave equation:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2_{\varrho,\alpha}\right]\times \Phi_{\varrho,\alpha}=0. \tag{54}$$

As solution set we select:

$$\Phi_{\varrho,\alpha}=e^{-\Gamma z}\Big[E'_{\varrho,\alpha}H_0^{(2)}(r_{\varrho,\alpha}r)+\sum_{i=0}^{\infty}E''_{\varrho,\alpha,i}J_i(r_{\varrho,\alpha}r)\cos(i\varphi)\Big].$$

This equation already considers the symmetry about $\varphi = 0$. /59
Because of the periodicity all terms where i is not divisible by four are eliminated in the sum, i.e.:

$$\Phi_{\varrho,\alpha}=e^{-\Gamma z}\Big[E'_{\varrho,\alpha}H_0^{(2)}(r_{\varrho,\alpha}r)+\sum_{i=0}^{\infty}E''_{\varrho,\alpha,4i}J_{4i}(r_{\varrho,\alpha}r)\cos(4i\varphi)\Big]. \tag{55}$$

Here, the amplitudes of the emitting field have a single prime, those of the scattering field a double prime. Equation 55 meets the wave equation

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+(k^2_{\varrho,\alpha}+\Gamma^2)-\frac{i^2}{r^2}\right]\Phi_{\varrho,\alpha}=0 \tag{56}$$

with

$$\varepsilon_{\varrho,\alpha}^2 = k_{\varrho,\alpha}^2 + \Gamma^2$$

and

$$\mathrm{Im}(\varepsilon_{\varrho,\alpha}) < 0 .$$

It also meets the conditions of axial symmetric excitation of the emitting field, emission condition, the periodicity properties of the scattering field.

Now the scattering field should be described as the sum of the emitting fields of other fibers, i.e., we should have:

$$\begin{aligned} &\sum_{i=0}^{\infty} E''_{4i} J_{4i}(\varepsilon r) \cos(4 i \varphi) = \\ &= E' \sum_{\substack{m,n \\ \neq 0,0}} H_0^{(2)}\left(\varepsilon \sqrt{r^2 + R_{m,n}^2 - 2 R_{mn} r \cos(\psi_{mn} - \varphi)}\right). \end{aligned} \tag{57}$$

Here we have:

$$\begin{aligned} R_{mn} &= d \sqrt{m^2 + n^2} , \\ \cos\psi_{mn} &= m / \sqrt{m^2 + n^2} , \\ \sin\psi_{mn} &= n / \sqrt{m^2 + n^2} . \end{aligned} \tag{58}$$

according to the Gegenbauer addition theorem, the right side of equation 57 is:

$$\begin{aligned} &E' \sum_{\substack{m,n \\ \neq 0,0}} [J_0(\varepsilon r) H_0(\varepsilon R_{mn}) + \\ &+ 2 \sum_{k=1}^{\infty} J_k(\varepsilon r) H_k(\varepsilon R_{mn}) \cos k(\psi_{mn} - \varphi)] = \\ &= E' J_0(\varepsilon r) \sum_{\substack{m,n \\ \neq 0,0}} H_0(\varepsilon R_{mn}) + \\ &+ 2 E' \sum_{k=1}^{\infty} J_k(\varepsilon r) \sum_{\substack{m,n \\ \neq 0,0}} H_k(\varepsilon R_{mn}) \times \\ &\times (\cos k \psi_{mn} \cos k \varphi + \sin k \psi_{mn} \sin k \varphi) . \end{aligned} \tag{59}$$

In equation 59 we know:

$$\sum_{\substack{m,n \\ \neq 0,0}} H_k(\varepsilon R_{mn}) \sin k\psi_{mn} \sin k\varphi = 0,$$

since:

$$\sin k\psi_{mn} = \sin \psi_{mn} \, [\text{Polynom in } \cos \psi_{mn}],$$

i.e. after summation, the terms with + n and - n cancel out.
In addition:

$$\sum_{\substack{m,n \\ \neq 0,0}} H_k(\varepsilon R_{mn}) \cos k\psi_{mn} \cos k\varphi = 0$$

for k = 1,3,5,...,
since cos (2i - 1) ψ_{mn} = cos ψ_{mn} [Polynom in $\sin^2 \psi_{mn}$]
i.e. terms with + m and -m cancel.
We have :

$$\cos i(2x) = 2^{i-1} \cos^i 2x - \frac{i}{1}(i-3)\, 2^{i-3} \cos^{i-2} 2x + \frac{i}{2}\binom{i-3}{1} \times 2^{i-5} \cos^{i-4} 2x - + \cdots,$$

where:

$$\cos 2x = \cos^2 x - \sin^2 x = (m^2 - n^2)/(m^2 + n^2).$$

If i is odd, i.e. 2i is not divisible by four, the terms cos i2x reverse sign upon exchange of m and n, then the sum is equal to zero.

If i is even, the signs are retained. So from equation 57:

$$\sum_{i=0}^{\infty} E''_{4i} J_{4i}(\varepsilon r) \cos 4i\varphi = E' J_0(\varepsilon r) \sum_{\substack{m,n \\ \neq 0,0}} H_0(\varepsilon R_{mn}) + 2E' \sum_{k=1}^{\infty} J_{4k}(\varepsilon r) \sum_{\substack{m,n \\ \neq 0,0}} H_{4k}(\varepsilon R_{mn}) \cos 4k\psi_{mn} \cos 4k\varphi \qquad (60)$$

and we immediately have, by coefficient comparison:

$$E''_{\varrho,\alpha,0} = E''_{\varrho,\alpha} \sum_{\substack{m,n \\ \neq 0,0}} H_0^{(2)}(\varepsilon_{\varrho,\alpha} R_{mn}), \qquad (61)$$

$$E''_{\varrho,\alpha,4i} = 2E'_{\varrho,\alpha} \sum_{\substack{m,n \\ \neq 0,0}} H_{4i}^{(2)}(\varepsilon_{\varrho,\alpha} R_{mn}) \cos 4i\psi_{mn}. \qquad (62)$$

The coefficients from equation 61 and 62 are substituted into equation

55 and we obtain:

$$\begin{aligned}\Phi_{\varrho,\alpha}(r,\varphi,z) = e^{-\Gamma z}\, E'_{\varrho,\alpha}\Big[& H_0^{(2)}(\varepsilon_{\varrho,\alpha} r) + \\ + & J_0(\varepsilon_{\varrho,\alpha} r) \sum_{\substack{m,n \\ \neq 0,0}} H_0^{(2)}(\varepsilon_{\varrho,\alpha} R_{mn}) + \\ + & 2\sum_{i=1}^{\infty} J_{4i}(\varepsilon_{\varrho,\alpha} r) \cos 4i\varphi \sum_{\substack{m,n \\ \neq 0,0}} H_{4i}^{(2)}(\varepsilon_{\varrho,\alpha} R_{mn}) \times \\ \times & \cos 4i\varphi_{mn}\Big].\end{aligned} \tag{63}$$

Thus, the two scalar potentials are determined, except for one amplitude and propagation constant.

7.3 Determination of the Vector Potential

We set the emitting fraction of the vector potential equal to that of the individual fiber:

$$\boldsymbol{A} = (0, A'_\varphi, 0)$$

with

$$A'_\varphi = E'_\nu H_1^{(2)}(\varepsilon_\nu r)\, e^{-\Gamma z}. \tag{64}$$

This equation is axial symmetric. /60

For the constant field the wave equations apply:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2} + k_\nu^2\right] \times A''_r - \frac{2}{r^2}\frac{\partial A''_\varphi}{\partial \varphi} = 0, \tag{65}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2} + k_\nu^2\right] \times A''_\varphi + \frac{2}{r^2}\frac{\partial A''_r}{\partial \varphi} = 0, \tag{66}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k_\nu^2\right] A''_z = 0. \tag{67}$$

Since according to our assumption, the scattering field is illustrated as a super-position of the emitting field, it follows that $A_z'' = 0$, since it cannot be excited by A'_φ.

(By vector addition of all emitting terms, only components in the r, φ plane can exist and no z- component).

As a general solution we select.

$$
\begin{aligned}
A_r''(r,\varphi,z) &= e^{-\Gamma z}\sum_i \{\sin i\varphi[A_i'' J_{i-1}(\varepsilon_\nu r) + \\
&\quad + a_i'' J_{i+1}(\varepsilon_\nu r)] + \\
&\quad + \cos i\varphi[B_i'' J_{i+1}(\varepsilon_\nu r) + b_i'' J_{i-1}(\varepsilon_\nu r)]\}, \qquad (68)\\
A_\varphi''(r,\varphi,z) &= e^{-\Gamma z}\sum_i \{\cos i\varphi[A_i'' J_{i-1}(\varepsilon_\nu r) - \\
&\quad - a_i'' J_{i+1}(\varepsilon_\nu r)] + \\
&\quad + \sin i\varphi[B_i'' J_{i+1}(\varepsilon_\nu r) - b_i'' J_{i-1}(\varepsilon_\nu r)]\}, \qquad (69)\\
A_z''(r,\varphi,z) &= 0. \qquad (70)
\end{aligned}
$$

For the generated speeds of sound (only the fraction of the constant field) we have:

$$v_r'' = \mathrm{rot}_r A'' = \Gamma A_\varphi'', \qquad (71)$$

$$v_\varphi'' = \mathrm{rot}_\varphi A'' = -\Gamma A_r'', \qquad (72)$$

$$v_z'' = \mathrm{rot}_z A'' = \frac{1}{r} A_\varphi'' + \frac{\partial}{\partial r} A_\varphi'' - \frac{1}{r}\frac{\partial}{\partial\varphi} A_r''. \qquad (73)$$

The field period is periodic in φ by $\pi/2$:

$$A''(\varphi + \pi/2) = A''(\varphi).$$

The field is symmetric at $\varphi = 0$ (and because of the periodicity, also at $\varphi = n\pi/2$).

Therefore we must have:

$$1) \quad v_\varphi(\varphi = 0) = 0$$

and thus

$$A_r''(\varphi=0)=0,$$

and from this we obtain:

$$B_i'' J_{i+1} + b_i'' J_{i-1} = 0,$$

from which follows:

$$B_0'' = b_0'',\quad B_i'' = b_i'' = 0 \qquad \text{for } i= 1,2,3,\ldots \tag{74}$$

2) Because of the periodicity, all terms where i is not divisible by four drop out.

So as an equation meeting the conditions of symmetry and periodicity we have:

$$A_r'' = e^{-\Gamma z}\sum_i \sin 4i\varphi\,[A_{4i}'' J_{4i-1}(\varepsilon_\nu r) + a_{4i}'' J_{4i+1}(\varepsilon_\nu r)], \tag{75}$$

$$A_\varphi'' = e^{-\Gamma z}\sum_i \cos 4i\varphi\,[A_{4i}'' J_{4i-1}(\varepsilon_\nu r) - a_{4i}'' J_{4i+1}(\varepsilon_\nu r)], \tag{76}$$

$$A_z'' = 0. \tag{77}$$

Equations 75 and 76 must still meet the divergence condition:

$$\operatorname{div} A'' = \frac{1}{r}A_r'' + \frac{\partial}{\partial r}A_r'' + \frac{1}{r}\frac{\partial}{\partial\varphi}A_\varphi'' \equiv 0,$$

$$\frac{1}{r}\left(A_r'' + \frac{\partial}{\partial\varphi}A_\varphi''\right) = e^{-\Gamma z}\frac{1}{r}\sum_i \sin 4i\varphi\,[A_{4i}'' J_{4i-1}(\varepsilon_\nu r)(1-4i) + a_{4i}'' J_{4i+1}(\varepsilon_\nu r)(1+4i)],$$

$$\frac{\partial}{\partial r}A_r'' = e^{-\Gamma z}\sum_i \sin 4i\varphi\,[A_{4i}''\varepsilon_\nu J_{4i-1}'(\varepsilon_\nu r) + a_{4i}''\varepsilon_\nu J_{4i+1}'(\varepsilon_\nu r)].$$

thus we have:

$$\operatorname{div} A'' = e^{-\Gamma z}\sum_i \Big\{\sin 4i\varphi\Big[\varepsilon_\nu A_{4i}''\Big(J_{4i-1}'(\varepsilon_\nu r) - \frac{4i-1}{\varepsilon_\nu r}J_{4i-1}(\varepsilon_\nu r)\Big) + \varepsilon_\nu a_{4i}''\Big(J_{4i+1}'(\varepsilon_\nu r) + \frac{4i+1}{\varepsilon_\nu r}J_{4i+1}(\varepsilon_\nu r)\Big)\Big]\Big\}.$$

because

$$J_p'(z) - \frac{p}{z} J_p(z) = -J_{p+1}(z) ,$$

$$J_p'(z) + \frac{p}{z} J_p(z) = J_{p-1}(z)$$

from this follows:

$$\operatorname{div} A'' = e^{-\Gamma z} \varepsilon_\nu \sum_i \sin 4 i \varphi J_{4i}(\varepsilon_\nu r) \times$$
$$\times (-A''_{4i} + a''_{4i}) = 0$$

and from this we obtain:

$$A''_{4i} = a''_{4i} . \qquad (78)$$

So as a solution we obtain the following expression which meets the condition of symmetry, periodicity and divergence:

$$A''_r = \frac{e^{-\Gamma z}}{\varepsilon_\nu r} 8 i A''_{4i} \sin 4 i \varphi J_{4i}(\varepsilon_\nu r) , \qquad (79)$$

$$A''_\varphi = e^{-\Gamma z} 2 A''_{4i} \cos 4 i \varphi J'_{4i}(\varepsilon_\nu r) , \qquad (80)$$ /61

$$A'_z = 0 . \qquad (81)$$

Now the constant field 79, 80 should originate as the superposition of the emitting field 64 of all individual fibers. These fibers are added vectorally.

The emitting field of the fiber with the indices m, n generates the component $\hat{A}'$ at the plotted point P. This is split apart in the coordinate system (r, φ) of the fiber with coordinates (0,0) into a radial component $\hat{A}''_r$ and an azimuth component $\hat{A}''_\varphi$.

From Figure 2 we can see:

$$\alpha = \beta - \pi/2 ,$$
$$\hat{A}''_r = \hat{A}' \cos\alpha = \hat{A}' \cos(\beta - \pi/2) = \hat{A}' \sin\beta ,$$
$$\hat{A}''_\varphi = -\hat{A}' \sin\alpha = -\hat{A}' \sin(\beta - \pi/2) = \hat{A}' \cos\beta ,$$
$$\sin\beta = (R_{mn}/u) \sin(\varphi_{mn} - \varphi) \ {}^{1}(\text{Sinussatz}) ,$$
$$u^2 = r^2 + R^2_{mn} - 2 r R_{mn} \cos(\varphi_{mn} - \varphi)$$
$${}^{2}(\text{Cosinussatz}) .$$

Key: 1-sine law 2-cosine law

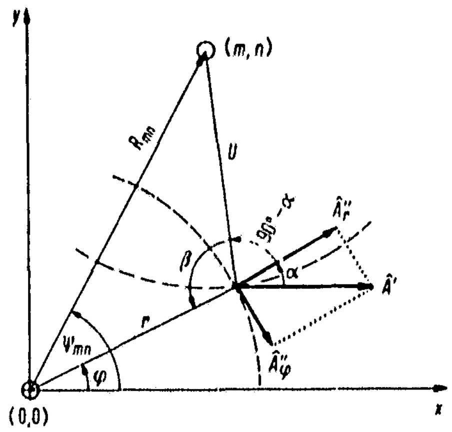

Figure 2: The addition of vector potentials of the fibers (m, n) at plotted point P.

From this follows:

$$\cos\beta = \sqrt{1-\sin^2\beta} = \frac{1}{u}\left|r - R_{mn}\cos(\psi_{mn}-\varphi)\right| .$$

Thus we have:

$$\hat{A}_r'' = \hat{A}' \frac{R_{mn}\sin(\psi_{mn}-\varphi)}{\sqrt{r^2+R_{mn}^2-2\,r\,R_{mn}\cos(\psi_{mn}-\varphi)}}, \tag{82}$$

$$\hat{A}_\varphi'' = \hat{A}' \frac{r - R_{mn}\cos(\psi_{mn}-\varphi)}{\sqrt{r^2+R_{mn}^2-2\,r\,R_{mn}\cos(\psi_{mn}-\varphi)}}\,. \tag{83}$$

thus:

$$\hat{A}' = e^{-\Gamma z} E_\nu' H_1^{(2)} \times \\ \times \left(\varepsilon_\nu \sqrt{r^2+R_{mn}^2-2\,r\,R_{mn}\cos(\psi_{mn}-\varphi)}\right). \tag{84}$$

Now we should have:

$$A_r'' = \sum_{\substack{m,n \\ \neq 0,0}} \hat{A}_r''\,; \qquad A_\varphi'' = \sum_{\substack{m,n \\ \neq 0,0}} \hat{A}_\varphi''$$

or for the radial component:

$$\frac{1}{\varepsilon_\nu r}\sum_i 8iA''_{4i}\sin(4i\varphi)J_{4i}(\varepsilon_\nu r) = \tag{85}$$
$$-E'_\nu\sum_{\substack{m,n\\ \neq 0,0}}\frac{R_{mn}\sin(\psi_{mn}-\varphi)}{\sqrt{r^2+R_{mn}^2-2rR_{mn}\cos(\psi_{mn}-\varphi)}}\times$$
$$\times H_1^{(2)}\left(\varepsilon_\nu\sqrt{r^2+R_{mn}^2-2rR_{mn}\cos(\psi_{mn}-\varphi)}\right).$$

and for the azimuth component:

$$2\sum_i A''_{4i}\cos 4i\varphi\, J'_{4i}(\varepsilon_\nu r) = \tag{86}$$
$$-E'_\nu\sum_{\substack{m,n\\ \neq 0,0}}\frac{r-R_{mn}\cos(\psi_{mn}-\varphi)}{\sqrt{r^2+R_{mn}^2-2rR_{mn}\cos(\psi_{mn}-\varphi)}}\times$$
$$\times H_1^{(2)}\left(\varepsilon_\nu\sqrt{r^2+R_{mn}^2-2rR_{mn}\cos(\psi_{mn}-\varphi)}\right).$$

According to the Gegenbauer addition theorem, from the right side of equation 85 we have:

$$\frac{2E'_\nu}{\varepsilon_\nu r}\sum_{k=0}^{\infty}(1+k)J_{1+k}(\varepsilon_\nu r)\sum_{\substack{m,n\\ \neq 0,0}}H^{(2)}_{1+k}(\varepsilon_\nu R_{mn})\times$$
$$\times\sin[(1+k)(\psi_{mn}-\varphi)] =$$
$$\frac{2E'_\nu}{\varepsilon_\nu r}\sum_{k=0}^{\infty}(1+k)J_{1+k}(\varepsilon_\nu r)\sum_{\substack{m,n\\ \neq 0,0}}\times \tag{87}$$
$$\times H^{(2)}_{1+k}(\varepsilon_\nu R_{mn})[\sin(1+k)\psi_{mn}\cos(1+k)\varphi -$$
$$-\cos(1+k)\psi_{mn}\sin(1+k)\varphi].$$

Like the scalar potentials (see equation 59ff) the terms with $\sin(1+k)\ \psi_{mn}$ and terms with $\cos(1+k)\ \psi_{mn}$, cancel where $(1+k)$ is not divisible by four.

With $(1+k) = 4i$ from equation 87 we have:

$$\frac{-2E'_\nu}{\varepsilon_\nu r}\sum_{i=0}^{\infty}4iJ_{4i}(\varepsilon_\nu r)\sin 4i\varphi\sum_{\substack{m,n\\ \neq 0,0}}H^{(2)}_{4i}\times$$
$$\times(\varepsilon_\nu R_{mn})\cos 4i\psi_{mn} \tag{88}$$

And the coefficient comparison between equation 88 and left side of equation 85 immediately yields:

$$A''_{4i} = E'_r \sum_{\substack{m,n \\ \neq 0,0}} H^{(2)}_{4i}(\varepsilon_r R_{mn}) \cos(4 i \psi_{mn}) \qquad (89)$$

$$i = 1, 2, 3, \ldots$$

A''_0 is not yet determined since the terms for i = 0 disappear on both sides.

From the right side of equation 86 according to the addition theorem we have: /62

$$\frac{2 E'_r}{\varepsilon_r r} \sum_{k=0}^{\infty} (1+k) J_{1+k}(\varepsilon_r r) \sum_{\substack{m,n \\ \neq 0,0}} H^{(2)}_{1+k}(\varepsilon_r R_{mn}) \times$$

$$\times \frac{r - R_{mn} \cos(\psi_{mn} - \varphi)}{R_{mn} \sin(\psi_{mn} - \varphi)} \sin(1+k)(\psi_{mn} - \varphi). \qquad (90)$$

A rearrangement which permits a direct coefficient comparison of r and φ is not possible. Since A''_0 is a term independent of angle, for A''_0 a coefficient comparison identical in r is sufficient for the angle independent fraction of equation 90 and the left side of equation 86. For this, only the term for k = 0 or i = 0 is taken from equation 90 and the left side of equation 86. Then we obtain:

$$-2 A''_0 J_1(\varepsilon_r r) = \frac{2 E'_r}{\varepsilon_r r} J_1(\varepsilon_r r) \sum_{\substack{m,n \\ \neq 0,0}} H^{(2)}_1(\varepsilon_r R_{mn}) \times$$

$$\times \frac{r - R_{mn} \cos(\psi_{mn} - \varphi)}{R_{mn}} =$$

$$= \frac{2 E'_r}{\varepsilon_r r} J_1(\varepsilon_r r) \sum_{\substack{m,n \\ \neq 0,0}} H^{(2)}_1(\varepsilon_r R_{mn}) \times$$

$$\times [r/R_{mn} - \cos\psi_{mn} \cos\varphi + \sin\psi_{mn} \sin\varphi].$$

The terms with cos ψ_{mn} disappear in the sum due to alternation of n with −n. We then have:

$$A''_0 = -E'_r \sum_{\substack{m,n \\ \neq 0,0}} \frac{H^{(2)}_1(\varepsilon_r R_{mn})}{\varepsilon_r R_{mn}}. \qquad (91)$$

From equations 64, 79, 80, 81, 89 and 91 we obtain for the vector potential A = A' + A":

$$
\begin{aligned}
A_r(r, \varphi, z) = & -\frac{2 E'_r e^{-\Gamma z}}{\varepsilon_r r} \times \\
& \times \sum_{i=1}^{\infty} \Big\{ 4 i \sin 4 i \varphi \, J_{4i}(\varepsilon_r r) \times \\
& \times \sum_{\substack{m,n \\ \neq 0,0}} H^{(2)}_{4i}(\varepsilon_r R_{mn}) \cos 4 i \varphi_{mn} \Big\} ,
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
A_\varphi(r, \varphi, z) = & \; E'_r e^{-\Gamma z} \Big\{ H^{(2)}_1(\varepsilon_r r) + 2 J_1(\varepsilon_r r) \times \\
& \times \sum_{\substack{m,n \\ \neq 0,0}} \frac{H^{(2)}_1(\varepsilon_r R_{mn})}{\varepsilon_r R_{mn}} - 2 \sum_{i=1}^{\infty} \cos(4 i \varphi) J'_{4i}(\varepsilon_r r) \times \\
& \times \sum_{\substack{m,n \\ \neq 0,0}} H^{(2)}_{4i}(\varepsilon_r R_{mn}) \cos 4 i \varphi_{mn} \Big\} .
\end{aligned}
\tag{93}
$$

Thus the vector potential is determined, except for one amplitude quantity.

7.4 Comments

For the constant field we have permitted an angle dependence, the emitting field of a fiber has been assumed to be axial symmetric. This is not entirely correct because the fields affect each other and an angle-dependent scattering field will always result in an angle-dependent emitting field. The solution presented can be considered to be an approximate solution. As a next step, we would have to proceed from an emitting field described by equation 63, 92, 93 and to perform perturbation calculations with equations 79 and 80. This is not possible either analytically or numerically. Within the framework of the achieved accuracy, we must assume that at least in the vicinity of the fibers an angle independent field exists. This is achieved by neglecting the last (angle dependent) sum in equations 63 and 93 and setting equation 92 equal to zero. This procedure is justified by consideration of the order of magnitude of the individual summands.

In equation 92 A_r disappears for $r \rightarrow 0$ on the order of $O(r^3)$, likewise the last sum in equation 93, whereas the first sum in equation 93 increases with 1/r.

To formulate the boundary conditions at the fiber surface we therefore use the following descriptions of the potential:

$$\Phi_{\varrho,\alpha}(r,\varphi,z) = E_{\varrho,\alpha} e^{-\Gamma z}\Big[H_0^{(2)}(\varepsilon_{\varrho,\alpha} r) + J_0(\varepsilon_{\varrho,\alpha} r) \sum_{\substack{m,n \\ \neq 0,0}} H_0^{(2)}(\varepsilon_{\varrho,\alpha} R_{mn})\Big], \tag{94}$$

$$A_\varphi(r,\varphi,z) = E_\nu e^{-\Gamma z}\Big[H_1^{(2)}(\varepsilon_\nu r) + 2J_1(\varepsilon_\nu r) \sum_{\substack{m,n \\ \neq 0,0}} \frac{H_1^{(2)}(\varepsilon_\nu R_{mn})}{\varepsilon_\nu R_{mn}}\Big], \tag{95}$$

$$A_r = A_z = 0.$$

The boundary conditions to be met are the same as for the individual fiber.

8. Integration Method

In the last section we illustrated the scattering field as the discreet sum of the emitting fields. We expressly used the regular arrangement of fibers. But in a real absorber this regularity does not exist. We could account for this by introducing an average fiber density and illustrating the scattering field as an integral over the emitting terms of the fibers distributed uniformly about an average.

Let N be the number of fibers per surface unit. Then g is the amount for one fiber and NgdF is the contribution of the surface element dF.

In Figure 3 we determine the contribution of the source region about Q at the plotted point P. We again use equations 42 to 44 for the individual fibers as the emitting field of the fibers: /63

$$\Phi_{\varrho,\alpha} = E'_{\varrho,\alpha} H_0^{(2)}(\varepsilon_{\varrho,\alpha} u)\, e^{-\Gamma z},$$

$$A'_\varphi = E'_\nu H_1^{(2)}(\varepsilon_\nu u)\, e^{-\Gamma z}.$$

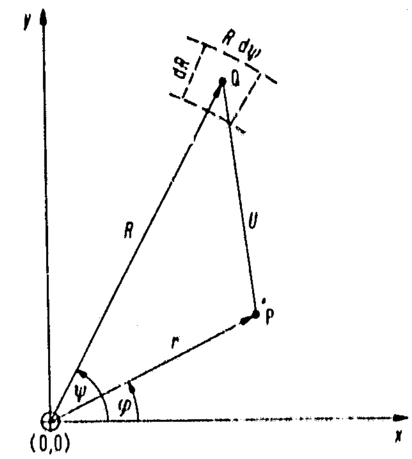

Figure 3: Integration relationships

The surface element becomes

$$\mathrm{d}F = R\,\mathrm{d}R\,\mathrm{d}\psi\,.$$

From this results the contribution of the source region about Q at the plotted point P for the scalar potential:

$$\mathrm{d}\Phi''_{\varrho,\alpha} = \mathrm{e}^{-\Gamma z} N\,R\,\mathrm{d}R\,\mathrm{d}\psi \times \\ \times E'_{\varrho,\alpha} H_0^{(2)}\left(\kappa_{\varrho,\alpha}\sqrt{r^2 + R^2 - 2\,r\,R\cos(\varphi - \psi)}\right)$$

and from this:

$$\Phi''_{\varrho,\alpha} = \int_0^{2\pi}\mathrm{d}\psi \int_d^{\infty} \mathrm{e}^{-\Gamma z} N\,R\,E'_{\varrho,\alpha} H_0^{(2)} \times \\ \times \left(\kappa_{\varrho,\alpha}\sqrt{r^2 + R^2 - 2\,r\,R\cos(\varphi - \psi)}\right)\mathrm{d}R\,. \quad (96)$$

According to the addition theorem:

$$H_0^{(2)}\left(\kappa_{\varrho,\alpha}\sqrt{r^2 + R^2 - 2\,r\,R\cos(\varphi - \psi)}\right) = \\ = J_0(\kappa_{\varrho,\alpha} r)\, H_0^{(2)}(\kappa_{\varrho,\alpha} R) + \\ + 2\sum_{n=1}^{\infty} J_n(\kappa_{\varrho,\alpha} r)\, H_0^{(2)}(\kappa_{\varrho,\alpha} R)\cos[n(\varphi - \psi)]\,.$$

Upon integration over ψ the terms of the sum over n cancel because of the cosine. Therefore from equation 96 we have:

$$\Phi''_{\varrho,\alpha} = e^{-\Gamma z} E'_{\varrho,\alpha} N\,J_0(\kappa_{\varrho,\alpha} r)\int_0^{2\pi}\mathrm{d}\psi \int_d^{\infty} R\,H_0^{(2)}(\kappa_{\varrho,\alpha} R)\,\mathrm{d}R$$

and from this:

$$\Phi''_{\varrho,\alpha} = -2\pi e^{-\Gamma z} E'_{\varrho,\alpha} N d^2 \frac{H_1^{(2)}(\varepsilon_{\varrho,\alpha} d)}{\varepsilon_{\varrho,\alpha} d} J_0(\varepsilon_{\varrho,\alpha} r). \tag{97}$$

For the vector potential we must perform vector addition. From the discussion on radial or azimuth components in equations 82, 83 and 84, we can immediately write the constant fraction:

$$\begin{aligned} A''_r(r,\varphi,z) &= \\ &= e^{-\Gamma z} E'_\nu N \int_{\varphi-\pi}^{\varphi+\pi} d\psi \int_d^\infty \frac{R^2 \sin(\psi-\varphi)}{\sqrt{r^2+R^2-2rR\cos(\psi-\varphi)}} \times \\ &\times H_1^{(2)}\left(\varepsilon_\nu \sqrt{r^2+R^2-2rR\cos(\psi-\varphi)}\right) dR, \end{aligned} \tag{98}$$

$$\begin{aligned} A''_\varphi(r,\varphi,z) &= \\ &= e^{-\Gamma z} E'_\nu N \int_{\varphi-\pi}^{\varphi+\pi} d\psi \int_d^\infty R \frac{r-R\cos(\psi-\varphi)}{\sqrt{r^2+R^2-2rR\cos(\psi-\varphi)}} \times \\ &\times H_1^{(2)}\left(\varepsilon_\nu \sqrt{r^2+R^2-2rR\cos(\psi-\varphi)}\right) dR. \end{aligned} \tag{99}$$

The integration limits with regard to ψ are variable as long as integration proceeds over a full circle. The selected shape permits a simple transformation: $\alpha = \psi - \varphi$. Thus from equations 98 and 99 we have:

$$\begin{aligned} A''_r &= e^{-\Gamma z} E'_\nu N \int_{-\pi}^{\pi} d\alpha \int_d^\infty \frac{R^2 \sin\alpha}{\sqrt{r^2+R^2-2rR\cos\alpha}} \times \\ &\times H_1^{(2)}\left(\varepsilon_\nu \sqrt{r^2+R^2-2rR\cos\alpha}\right) dR, \end{aligned} \tag{100}$$

$$\begin{aligned} A''_\varphi &= e^{-\Gamma z} E'_\nu N \int_{-\pi}^{\pi} d\alpha \int_d^\infty R \frac{r-R\cos\alpha}{\sqrt{r^2+R^2-2rR\cos\alpha}} \times \\ &\times H_1^{(2)}\left(\varepsilon_\nu \sqrt{r^2+R^2-2rR\cos\alpha}\right) dR. \end{aligned} \tag{101}$$

In equation 100 the integrand with respect to α is an odd function, i.e., $A''_r = 0$.

In equation 101 the integrand with respect to α is an even function.

According to the addition theorem we have:

$$H_1^{(2)}\left(\varepsilon_\nu \sqrt{r^2 + R^2 - 2\,r\,R\cos\alpha}\right) =$$
$$= \frac{2}{\varepsilon_\nu\, r\, R}\sum_{k=0}^{\infty}(1+k)\,J_{1+k}(\varepsilon_\nu r)\,H_{1+k}(\varepsilon_\nu R) \times \frac{\sin(1+k)\alpha}{\sin\alpha}\,\sqrt{r^2 + R^2 - 2\,r\,R\cos\alpha} \quad (102)$$

And thus from equation 101:

$$A_\varphi'' = \frac{4\,e^{-\Gamma z}E_\nu' N}{\varepsilon_\nu r}\int_d^\infty\int_0^\pi (r - R\cos\alpha)\times \quad (103)$$
$$\times \sum_{k=1}^{\infty} k\,J_k(\varepsilon_\nu r)\,H_k^{(2)}(\varepsilon_\nu R)\,\frac{\sin k\alpha}{\sin\alpha}\,\mathrm{d}\alpha\,\mathrm{d}R\,.$$

In equation 103, if $|\varepsilon_\nu r| \ll 1$, then $|J_k(\varepsilon_\nu r)|$ very quickly goes to zero with increasing k. In addition, the term $\sin k\alpha/\sin\alpha$ oscillates with increasing k more and more so that upon integration, only a slight contribution is provided. Of the sum appearing in the integrands we therefore consider only the term with k = 1. We then have:

$$A_\varphi'' = \frac{4\,e^{-\Gamma z}E_\nu' N}{\varepsilon_\nu r}\int_d^\infty\int_0^\pi (r - R\cos\alpha)\times$$
$$\times\; J_1(\varepsilon_\nu r)\,H_1^{(2)}(\varepsilon_\nu R)\,\mathrm{d}\alpha\,\mathrm{d}R\,.$$

Upon integration over α the first term of r – R cos α yields the factor π, the second term disappears. It becomes: /64

$$A_\varphi'' = 4\pi\,e^{-\Gamma z}E_\nu' N\,\frac{J_1(\varepsilon_\nu r)}{\varepsilon_\nu}\int_d^\infty H_1^{(2)}(\varepsilon_\nu R)\,\mathrm{d}R$$

or:

$$A_\varphi'' = 4\pi\,e^{-\Gamma z}E_\nu' N\,d^2\,\frac{H_2^{(2)}(\varepsilon_\nu d)}{(\varepsilon_\nu d)^2}\,J_1(\varepsilon_\nu r)\,. \quad (104)$$

Thus we obtain a description of the potentials:

$$\Phi_{\varrho,\alpha}(r,z) = E_{\varrho,\alpha}\,e^{-\Gamma z}\Big[H_0^{(2)}(\varepsilon_{\varrho,\alpha} r) -$$
$$- 2\pi N\,d^2\,\frac{H_1^{(2)}(\varepsilon_{\varrho,\alpha} d)}{\varepsilon_{\varrho,\alpha} d}\,J_0(\varepsilon_{\varrho,\alpha} r)\Big]\,, \quad (105)$$
$$A_\varphi(r,z) = E_\nu\,e^{-\Gamma z}\Big[H_1^{(2)}(\varepsilon_\nu r) +$$
$$+ 4\pi N\,d^2\,\frac{H_0^{(2)}(\varepsilon_\nu d)}{(\varepsilon_\nu d)^2}\,J_1(\varepsilon_\nu r)\Big]\,, \quad (106)$$
$$A_r = A_z \equiv 0\,.$$

With these descriptions the boundary conditions at the fiber surface are again fulfilled. They correspond to equations 94 and 95 for the model with quadratic, regular fiber arrangement and are comparable with these equations.

We have now described the fields of the compressional, temperature and viscosity waves in the environment of a fiber. Both in the complete field equations 63, 92, 93 as well as in the simplified equations 94, 95 for a regular quadratic fiber arrangement and in equations 105, 106 for a homogeneous fiber distribution, the amplitudes $E_{\rho,\alpha,\nu}$ and the axial propagation constant Γ are unknown. These are, as already shown in Section 6 for the individual fibers, determined through the boundary conditions of equations 33, 34, 35 at the surface of the reference fiber (m, n) = (0, 0), i.e., at $r = a$.

From a comparison of the equations 105 and 106 by the integration method with the results of the summation for regular fiber arrangement we now know that the component $A_r''(r, \varphi, z)$ of the scattering field quite disappears. This provides an additional argument for the simplification performed in Section 7.4, mainly, instead of using equation 92, to simply set $A_r(r, \varphi, z) = 0$. We also find a relation between equations 94, 95 and equations 105, 106:

$$\sum_{\substack{m,n \\ \neq 0,0}} H_0^{(2)}(r_{\varrho,\alpha} R_{mn}) \triangleq -2\pi N d^2 \frac{H_0^{(2)}(r_{\varrho,\alpha} d)}{r_{\varrho,\alpha} d}$$

$$\sum_{\substack{m,n \\ \neq 0,0}} \frac{H_1^{(2)}(r_\nu R_{mn})}{r_\nu R_{mn}} \triangleq 2\pi N d^2 \frac{H_0^{(2)}(r_\nu d)}{(r_\nu d)^2} .$$

In the complete field equations 63, 92, 93 we see in the last terms the influence of the spacial arrangement of neighboring fibers on the total field in the vicinity of a reference fiber. From the derivation of these terms as well as by its form, the influence of the order of symmetry is seen. If this increases, then in accordance with the factor in front of the summation index i, i.e., the order, of the first cylinder function increases in the terms and the interval between the orders becomes larger. The sums in the terms converge more quickly.

At this point we can account for the microstructure of the real absorber by introducing the statistically most frequent order of symmetry and by performing the summation of scattering fields of neighboring fibers only up to a certain interval and by applying the integration method to more distant fibers.

In part 2 of this work we will utilize the randomness of symmetry selection to derive a more simplified model.

10. Conclusions

One objective of absorber theory is the obtainment of a calculation method for the propagation constants and for the wave resistance of the absorber. Another objective is to gain knowledge about the influences of the individual structural parameters and material data on these values. Both objectives will be pursued in part 2.

Thanks are due the German Research Society at this point for financial support for the work.

(received for publication on March 4, 1976)

REFERENCES

1. Mechel, F., Royar, J., "Experimental Studies on the Theory of Porous Absorbers." Acustica 26 /1972/

2. Sides, D. J., Attenborough, K. and Mulholland, K. A., Application of a generalized acoustic propagation theory to fibrous absorbents. J. Sound Vib. 19 /1971/, 49.

3. Attenbrough, K., The influence of microstructure on propagation in porous fibrous absorbents. J. Sound Vib. 16 /1971/, 419.